Spontaneous symmetry breaking of (1+1)-dimensional ϕ^4 theory in light-front field theory (III)

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Abstract

We investigate (1+1)-dimensional ϕ^4 field theory in the symmetric and broken phases using discrete light-front quantization. We calculate the perturbative solution of the zero-mode constraint equation for both the symmetric and broken phases and show that standard renormalization of the theory yields finite results. We study the perturbative zero-mode contribution to two diagrams and show that the light-front formulation gives the same result as the equal-time formulation. In the broken phase of the theory, we obtain the nonperturbative solutions of the constraint equation and confirm our previous speculation that the critical coupling is logarithmically divergent. We discuss the renormalization of this divergence but are not able to find a satisfactory nonperturbative technique. Finally we investigate properties that are insensitive to this divergence, calculate the critical exponent of the theory, and find agreement with mean field theory as expected.

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I. Introduction

This paper is the third in a series of studies investigating one particular long range phenomenon, spontaneous symmetry breaking, in a particularly simple theory, ϕ^4 theory in (1+1)-dimensions. The Z_2 symmetry $\phi \rightarrow -\phi$ of the theory is spontaneously broken for a range of mass and coupling. At a critical coupling, this theory develops spontaneously new interactions that do not respect the Z_2 symmetry. In light-front field theory, this symmetry breaking is produced by the zero mode of the field. It can be shown that this zero mode is not an independent degree of freedom in the theory but depends on all the other degrees of freedom through a nonlinear operator constraint equation. In previous papers, which we will refer to as I [1] and II [2], we have shown how to solve this operator-valued constraint equation numerically and, in lowest order, analytically. We have investigated spontaneous symmetry breaking in low-order numerical calculations and discussed many of the properties of a spontaneously broken theory, including the critical coupling and tunnelling. Here we conclude this work with a perturbative analysis and higher order numerical calculations.

We provide a review of notation and basic ideas in the next section. In Sec. III, we discuss the perturbative behavior of the theory. We investigate the role of the zero mode in the perturbative regime and show that the usual mass subtraction renders the theory finite to all orders even when the zero-mode interactions are included. We also investigate the role of the zero-mode interactions in two loop diagrams. In Sec. IV we study the nonperturbative structure of the theory. We also look at the behavior of the matrix elements of the zero modes and calculate the critical coupling using a longitudinal momentum cutoff. Finally, we discuss the possible renormalization schemes for the broken phase of the theory. A summary of our findings is given in Sec. V.

II. Review

The details of the Dirac-Bergmann prescription and its application to the system considered in this paper are discussed elsewhere in the literature [1, 3, 4, 5]. In this section, we will summarize those results, introduce our notation, and review some of the results of I and II. We define light-front coordinates $x^\pm = \left(x^0 \pm x^1\right)/\sqrt{2}$. For a

classical field the $(\phi^4)_{1+1}$ Lagrangian is

$$\mathcal{L} = \partial_+\phi\partial_-\phi - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 . \tag{2.1}$$

We put the system in a box of length d and impose periodic boundary conditions. That is, we use "discretized light-cone quantization" (DLCQ) to regulate the system [6, 7]. Following the Dirac-Bergmann prescription, we can identify first-class constraints which define the conjugate momenta and a secondary constraint which determines the "zero mode" in terms of the other modes of the theory. The latter result can also be obtained by integrating the equations of motion in position space or differentiating the Hamiltonian with respect to the zero mode [1].

Quantizing, we define creation and annihilation operators $a_k^\dagger$ and $a_k = a_{-k}^\dagger$, along with the zero-mode operator a_0, by

$$\phi(x) = \frac{a_0}{\sqrt{4\pi}} + \frac{1}{\sqrt{4\pi}}\sum_{n\neq 0}\frac{a_n(x^+)}{\sqrt{|n|}}e^{ik_n^+x^-} , \tag{2.2}$$

where $k_n^+ = 2\pi n/d$ and summations run over all integers unless otherwise noted. These operators satisfy the usual commutation relations

$$[a_k, a_l] = 0 , \quad \left[a_k^\dagger, a_l^\dagger\right] = 0 , \quad \left[a_k, a_l^\dagger\right] = \delta_{k,l} , \quad k, l > 0 . \tag{2.3}$$

The light-front momentum operator P^+ is then given by

$$P^+ = \frac{2\pi}{d}\sum_{k>0}ka_k^\dagger a_k . \tag{2.4}$$

The quantum Hamiltonian that we use [1] is obtained from the light-front time evolution operator P^- by a rescaling

$$
\begin{aligned}
H &= \frac{96\pi^2}{\lambda d}P^- \\
&= \frac{g}{2}a_0^2 + \frac{a_0^4}{4} + g\Sigma_2 + 6\Sigma_4 \\
&\quad + \frac{1}{4}\sum_{n\neq 0}\frac{1}{|n|}\left(a_0^2 a_n a_{-n} + a_n a_{-n}a_0^2 + a_n a_0^2 a_{-n}\right.\\
&\quad \left. + a_n a_0 a_{-n}a_0 + a_0 a_n a_0 a_{-n} + a_0 a_n a_{-n}a_0 - 3a_0^2\right) \\
&\quad + \frac{1}{4}\sum_{k,l,m\neq 0}\frac{\delta_{k+l+m,0}}{\sqrt{|klm|}}(a_0 a_k a_l a_m + a_k a_0 a_l a_m + a_k a_l a_0 a_m + a_k a_l a_m a_0) \\
&\quad - C .
\end{aligned}
\tag{2.5}
$$

where $g = 24\pi\mu^2/\lambda$ and Σ_n is given by

$$\Sigma_n = \frac{1}{n!} \sum_{i_1,i_2,\ldots,i_n \neq 0} \delta_{i_1+i_2+\cdots+i_n,0} \frac{: a_{i_1} a_{i_2} \cdots a_{i_n} :}{\sqrt{|i_1 i_2 \cdots i_n|}} . \tag{2.6}$$

General arguments suggest that the Hamiltonian should be symmetric ordered [8]. However, it is not clear how one should treat the zero mode since it is not a dynamical field. As an *ansatz* we treat a_0 as an ordinary field operator when symmetric ordering the Hamiltonian. We have removed tadpoles by normal ordering the third and fourth terms and subtracting

$$\frac{3}{4} a_0^2 \sum_{n \neq 0} \frac{1}{|n|} . \tag{2.7}$$

In addition, we have subtracted a constant C so that the vacuum expectation value of H is zero. Note that this renormalization prescription is equivalent to a conventional mass renormalization and does not introduce any new operators (aside from the constant) into the Hamiltonian.

The constraint equation for the zero mode can be determined by taking a derivative of P^- with respect to a_0.

$$0 = ga_0 + a_0^3 + 2a_0\Sigma_2 + 2\Sigma_2 a_0 + \sum_{n>0} \frac{1}{n} \left(a_n^\dagger a_0 a_n + a_n a_0 a_n^\dagger - a_0 \right) + 6\Sigma_3 . \tag{2.8}$$

Using the constraint equation, we can rewrite H as:

$$\begin{aligned}
H &= g\Sigma_2 + 6\Sigma_4 - \frac{a_0^4}{4} + \frac{1}{4} \sum_{n \neq 0} \frac{1}{|n|} \left(a_n a_0^2 a_{-n} - a_0 a_n a_{-n} a_0 \right) \\
&\quad + \frac{1}{4} \sum_{k,l,m \neq 0} \frac{\delta_{k+l+m,0}}{\sqrt{|klm|}} \left(a_k a_0 a_l a_m + a_k a_l a_0 a_m \right) - C .
\end{aligned} \tag{2.9}$$

In light-front field theory the vacuum of the full theory is the perturbative Fock space vacuum. In I and II, using $(\phi^4)_{1+1}$ as an example, we have shown that the zero mode, which satisfies an operator-valued constraint equation, produces the long range physics of the theory, including spontaneous breaking of the Z_2 symmetry. We have found that the constraint equation can be solved using a Tamm–Dancoff truncation of the Fock space. For the one-mode truncation, we found a critical coupling consistent with the best equal-time calculations. Increasing the Tamm–Dancoff type truncation, we found rapid convergence with the total number of particles N. This is to be

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contrasted with the equal-time approach where an infinite number of particles are required to produce a critical point.

We found that above the critical coupling the zero mode develops a contribution that breaks the Z_2 symmetry of the theory. There are two such solutions to the constraint equation: one with $\langle 0|\phi|0\rangle > 0$ and one with $\langle 0|\phi|0\rangle < 0$. This, in turn, gives rise to two Hamiltonians which have equivalent spectra. The spectrum has the expected behavior: the Fock vacuum is the state of lowest energy and the lowest eigenvalue has a minimum at the critical coupling. Closer inspection of the field shows that tunneling occurs between positive and negative eigenvalues in the broken phase.

Our renormalization prescription removes tadpoles from ordinary interaction terms and would properly renormalize the theory if the zero mode were removed. However, inclusion of the zero mode appears to produces divergences in the constraint equation and in the resulting Hamiltonian. As a consequence, g_{critical} appears to grow logarithmically with the number of modes.

For $(\phi^4)_{1+1}$ we believe that a vanishing of the mass gap is associated with the critical coupling [9]. We have see that the gap between the vacuum and the lowest energy excited state is minimized at the critical coupling. One could imagine that, in the limit of large number of modes, the gap between the vacuum and the first excited state goes to zero at the critical coupling.

III.   Perturbation Theory

Even in weak coupling one must include the zero mode to correctly formulate a light-front quantized field theory. In this phase the zero-mode constraint equation can be solved perturbatively, and the theory that results when one substitutes the perturbative solution for the zero mode back into the Hamiltonian is very complicated. Nevertheless, one can systematically solve this theory order by order in perturbation theory.

There are two important questions that one must ask about the solutions: is the theory finite and does it agree with equal-time perturbation theory? Recall that in a pure ϕ^4 theory without the zero mode, the only infinities are tadpoles, and these have been removed with a mass counterterm (via normal ordering). Including the

zero mode in the Hamiltonian introduces new interactions that could lead to new infinities. Comparison of perturbative results with those of equal-time perturbation theory must be done because this is the only way we have of checking that the zero modes have not generated a completely different theory.

To address the renormalization problem we consider the Hamiltonian given in Eq. (2.9) and divide it into two parts

$$H = (g\Sigma_2 + 6\Sigma_4 - C) + H_{\text{zm}} . \tag{3.1}$$

The first part contains no zero modes; it is normal ordered and therefore totally finite. H_{zm} contains all of the zero-mode interactions

$$
\begin{aligned}
H_{\text{zm}} &= -\frac{a_0^4}{4} - \frac{1}{4} \sum_{n \neq 0} \frac{1}{|n|} \left(a_n a_0^2 a_{-n} - a_0 a_n a_{-n} a_0 \right) \\
&\quad + \frac{1}{4} \sum_{k,l,m \neq 0} \frac{\delta_{k+l+m,0}}{\sqrt{|klm|}} \left(a_k a_0 a_l a_m + a_k a_l a_0 a_m \right) .
\end{aligned}
\tag{3.2}
$$

The perturbative solution of the constraint equation (2.8) in $1/g$ is

$$a_0 = -\frac{6}{g}\Sigma_3 + \frac{6}{g^2} \sum_{n \neq 0} \frac{1}{|n|} \left(\Sigma_3 a_n a_{-n} + a_n a_{-n} \Sigma_3 + a_n \Sigma_3 a_{-n} - \frac{3\Sigma_3}{2} \right) + \dots . \tag{3.3}$$

We want to argue that H_{zm}, with a_0 given by Eq. (3.3), is finite. In two dimensions the only divergence comes from the self-contraction of a line: a tadpole. Notice that H_{zm} is made of the product of momentum conserving interactions. It contains a two point interaction of the form $a_n a_{-n}/|n|$, a three point interaction of the form $a_k a_l a_m \delta_{k+l+m,0}/\sqrt{|klm|}$, and the interactions in a_0. The operator a_0 itself contains the same two and three point interactions. We can then consider a general process in perturbation theory using these vertices.

The self-contraction of two lines associated with a three particle vertex would leave the third leg of the vertex with zero momentum. Since all legs are required to have a non-zero momentum we conclude that the three particle vertices cannot produce tadpoles. The explicit two-particle interaction in H_{zm} does give a divergence of the form
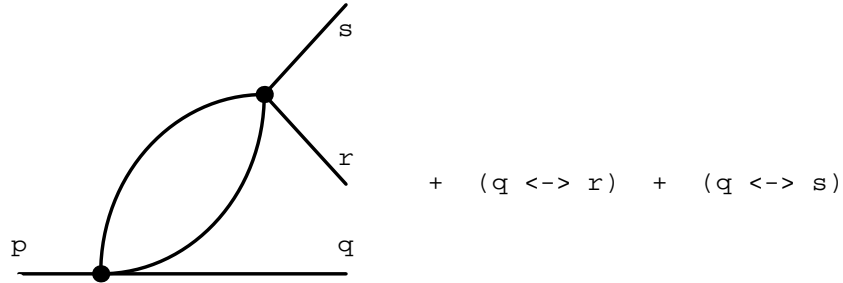
$$a_0{}^2 \sum_{n \neq 0} \frac{1}{|n|} . \tag{3.4}$$

Figure 1: The second order contribution to the 1 particle to 3 particle matrix element $[1 \to 3]$.

However there are two such interactions in H_{zm} and they cancel. The only remaining source of divergences are in the perturbative expansion of a_0. We see terms in the second order expansion of a_0 that have divergent tadpoles of the form

$$\Sigma_3 \sum_{n \neq 0} \frac{1}{|n|} , \qquad (3.5)$$

but again they cancel among themselves. This story is repeated order by order in perturbation theory, leaving a totally finite result to all orders.

Let us take a closer look at the unbroken-phase zero mode in perturbation theory. We know that the broken-phase zero mode produces spontaneous symmetry breaking in the Hamiltonian. What is the effect of the unbroken phase zero mode? For the unbroken phase, the contribution to loop integrals of the zero mode should vanish in the infinite volume limit, giving a "measure zero" contribution. For a box of finite volume d, the zero mode does contribute, compensating for the fact that the longest wavelength mode has been removed from the system. Thus, inclusion of the zero mode improves convergence to the infinite volume limit [10]; it acts as a form of infrared renormalization.

In addition, one can use the perturbative expansion of the zero mode to study the operator ordering problem. One can directly compare our operator ordering ansatz with a truly Weyl-ordered Hamiltonian and with Maeno's operator ordering prescription [11].

As an example, let us examine $O(\lambda^2)$ contributions to the processes $1 \to 3$ in Fig. 1 and $1 \to 1$ in Fig. 2. First, we calculate the "no zero mode" contributions. For

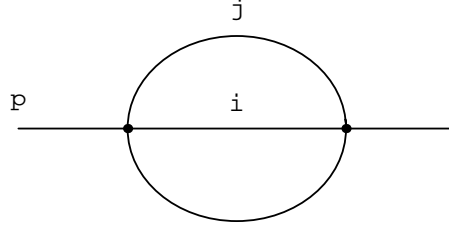Figure 2: The second order contribution to the propogator $[1 \to 1]$.

$1 \to 3$, we have

$$\langle 0|a_p \left(6\Sigma_4\right) \frac{1}{E - g\Sigma_2} \left(6\Sigma_4\right) a_q^\dagger a_r^\dagger a_s^\dagger |0\rangle =$$

$$\frac{18\delta_{p,q+r+s}}{g\sqrt{pqrs}} \sum_{k=1}^{r+s-1} \frac{1}{k\left(r+s-k\right)} \frac{1}{\frac{E}{g} - \frac{1}{q} - \frac{1}{k} - \frac{1}{r+s-k}} +$$

$$\left(q \leftrightarrow r\right) + \left(q \leftrightarrow s\right), \tag{3.6}$$

and for $1 \to 1$, we have

$$\langle 0|a_p \left(6\Sigma_4\right) \frac{1}{E - g\Sigma_2} \left(6\Sigma_4\right) a_p^\dagger |0\rangle =$$

$$\frac{6}{gp} \sum_{i,j>0}^{i+j<p} \frac{1}{ij\left(p-i-j\right)} \frac{1}{\frac{E}{g} - \frac{1}{i} - \frac{1}{j} - \frac{1}{p-i-j}}, \tag{3.7}$$

where E is the unperturbed energy. In the large volume limit $d \to \infty$, for fixed external momenta (in terms of P^+) and $E \propto 1/d$ (recall that H is a rescaled Hamiltonian), the sums can be converted to integrals. For $1 \to 3$, one obtains

$$-\frac{18\delta_{p,q+r+s}}{g\sqrt{pqrs}} \frac{4 \arctan \sqrt{\frac{\varepsilon}{4-\varepsilon}}}{\sqrt{\left(4 - \varepsilon\right)\varepsilon}} + \left(q \leftrightarrow r\right) + \left(q \leftrightarrow s\right), \tag{3.8}$$

with

$$\varepsilon = \left(\frac{E}{g} - \frac{1}{q}\right)\left(r + s\right), \tag{3.9}$$

where we restrict $\varepsilon < 4$. Likewise, for $1 \to 1$, one finds the large volume limit

$$-\frac{6}{gp} \frac{\pi^2}{4}, \tag{3.10}$$

where we have chosen $E = g/p$. It is straightforward to show that equal-time perturbation theory diagrams give the same result. If one computes the k^- integral in

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the equal-time formulation of the theory one finds a direct connection between the diagrams obtained in equal-time quantization and those found in light-front quantized field theory. This connection is particularly simple for scalar field theory in two dimensions [12].

Now we calculate the associated zero-mode contributions. We work to leading order in $1/g$. Substitution of the perturbative expansion (3.3) for a_0 in the Hamiltonian (3.2) produces an $O(g^{-1})$ term

$$H_{\text{zm}} = -\frac{6}{4g} \sum_{k,l,m \neq 0} \frac{\delta_{k+l+m,0}}{\sqrt{|klm|}} \left(a_k \Sigma_3 a_l a_m + a_k a_l \Sigma_3 a_m \right) \tag{3.11}$$

$$= -\frac{18 \Sigma_3 \Sigma_3}{g} - \frac{12}{g} \sum_{k,l>0} \frac{a_l^\dagger a_l}{kl \, (k+l)} + \frac{6}{g} \sum_{k,l>0} \frac{a_{k+l}^\dagger a_{k+l}}{kl \, (k+l)}$$

$$- \frac{3}{g} \sum_{k,l>0} \frac{1}{kl \, (k+l)} + O\left(g^{-2}\right) . \tag{3.12}$$

Thus, the contribution of the zero mode to the two processes is

$$\langle 0 | a_p H_{\text{zm}} a_q^\dagger a_r^\dagger a_s^\dagger | 0 \rangle = -\frac{18 \delta_{p,q+r+s}}{g \sqrt{pqrs}} \frac{1}{r+s} + (q \leftrightarrow r) + (q \leftrightarrow s) \tag{3.13}$$

for $1 \to 3$ and

$$\langle 0 | a_p H_{\text{zm}} a_p^\dagger | 0 \rangle = -\frac{9}{gp} \sum_{k=1}^{p-1} \frac{1}{k \, (p-k)} - \frac{12}{gp} \sum_{k>0} \frac{1}{k \, (k+p)} + \frac{6}{gp} \sum_{k=1}^{p-1} \frac{1}{k \, (p-k)} \tag{3.14}$$

for $1 \to 1$. We have discarded the constant term.

How does the zero mode affect convergence to the $d \to \infty$ limit? Look at Eq. (3.6). One can take the zero-momentum limit for each of the internal lines; that is, the limit $k \to 0$ plus the limit $k \to r+s$. The result is exactly *twice* Eq. (3.13). Likewise, for Eq. (3.7), one can take limits $i \to 0$ plus $j \to 0$ plus $i+j \to p$. The result is

$$-\frac{18}{gp} \sum_{k=1}^{p-1} \frac{1}{k \, (p-k)} , \tag{3.15}$$

exactly *twice* the first term in Eq. (3.14). (Note that the remaining terms of Eq. (3.14) are due to our particular operator ordering ansatz.) The appearance of this factor of two was also noted by Maeno [11]. As shown in Figs. 3 and 4, the zero mode greatly improves convergence to the large volume limit. The zero mode compensates, in an
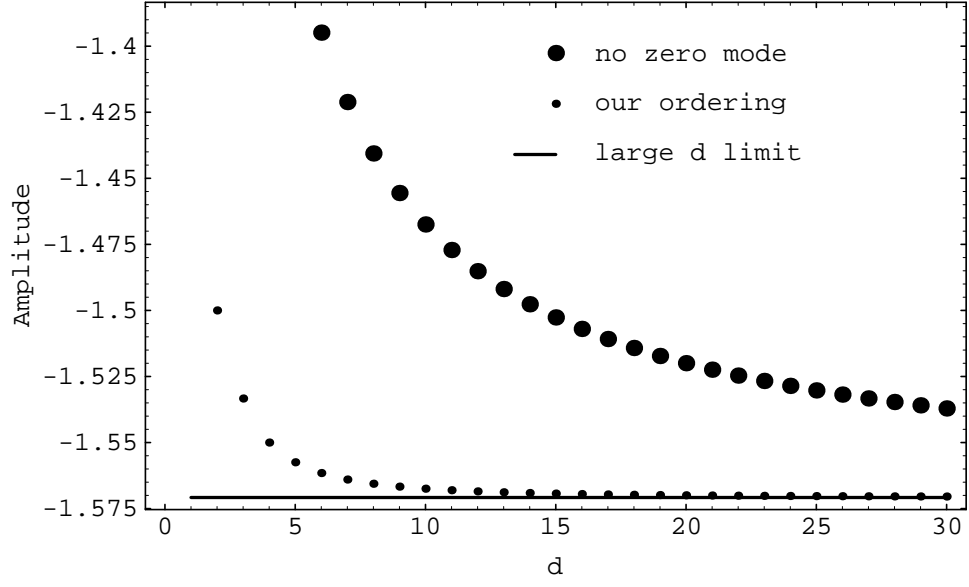
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Figure 3: Convergence to the large d limit of $1 \to 3$ with $\varepsilon = 2$.
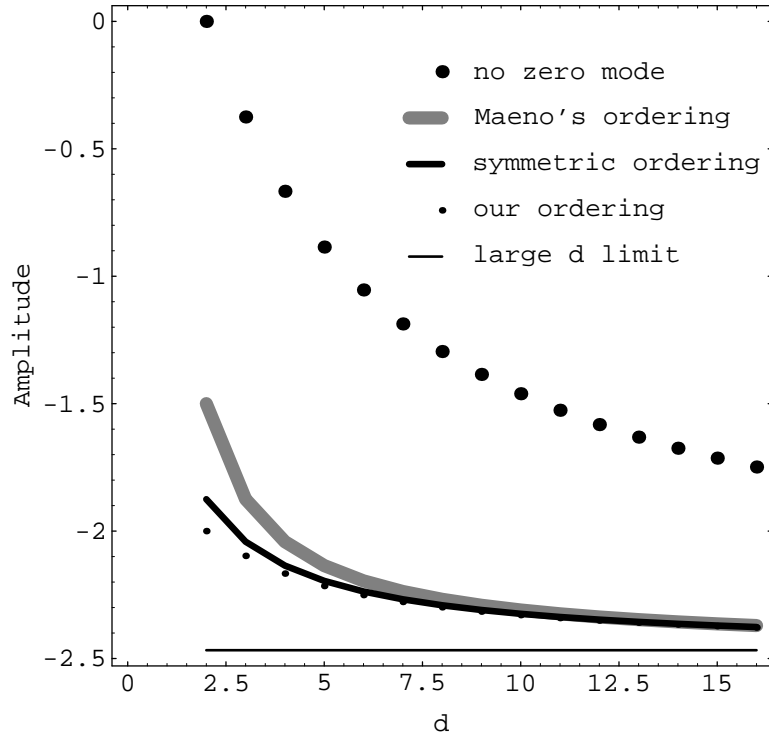

Figure 4: Convergence to the large d limit of $1 \to 1$ setting $E = g/p$ and dropping any constant terms.

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optimal manner, for the fact that we have removed the longest wavelength mode from the system.

One characteristic of our operator ordering ansatz is that the Hamiltonian, written in terms of the dynamical variables, is no longer symmetric ordered. That is, Eq. (3.11) is not symmetric ordered. However, in the context of perturbation theory, we are free to impose symmetric ordering on the Hamiltonian, order by order in $1/g$. Symmetric ordering Eq. (3.11) gives

$$-\frac{18\Sigma_3\Sigma_3}{g} - \frac{9}{g}\sum_{k,l>0}\frac{a_l^\dagger a_l}{kl\,(k+l)} + \frac{9}{2g}\sum_{k,l>0}\frac{a_{k+l}^\dagger a_{k+l}}{kl\,(k+l)}$$
$$-\frac{9}{4g}\sum_{k,l>0}\frac{1}{kl\,(k+l)} + O\!\left(g^{-2}\right) . \qquad (3.16)$$

Comparing with Eq. (3.12), we see that our operator ordering ansatz and true symmetric ordering produce the same basic operator structure with somewhat different coefficients. Finally, applying Maeno's operator ordering prescription to Eq. (3.11) produces simply

$$-\frac{18}{g}\Sigma_3\Sigma_3 . \qquad (3.17)$$

The relative effect of the three operator orderings on the $1 \to 1$ diagram is illustrated in Fig. 4. The difference between the different orderings is slight and decreases quickly with increasing d.

One can also look for the effect of the perturbative zero modes at tree level. The simplest process where this occurs is in the second order contributions to the six-point amplitude. There is a physical region where zero momentum is transferred in the line connecting the two vertices. If one omits the zero mode from this propagator the contribution of the diagram will be zero. Adding the zero mode leads to a more appealing non-zero result. If we were to compare the calculation of such a six point amplitude with experiment, we would get conflicting results since our zero momentum bin would give zero and the experiment would find some non-zero result. To avoid this apparent conflict one must use theoretical binning that is much finer than the experimental resolution. Then the null zero-mode contribution to lowest momentum experimental bin is small and the issue becomes one of convergence rates, which we have already addressed.

We can also consider making a perturbation expansion in the region where g is

negative. Following Robertson [13] one can construct an expansion in powers of $\sqrt{-g}$

$$a_0 = C_{-1}\sqrt{-g} + C_0 + \frac{C_1}{\sqrt{-g}} + \frac{C_2}{(-g)} + \frac{C_3}{(-g)^{3/2}} + \ldots \, . \tag{3.18}$$

We can substitute this expansion into the constraint equation and solve order by order for the operator coefficients C_n. In addition to the unbroken-phase solution we find:

$$
\begin{align}
C_{-1} &= 1 \text{ or } -1 \tag{3.19} \\
C_0 &= 0 \tag{3.20} \\
C_1 &= -3C_{-1}\Sigma_2 \tag{3.21} \\
C_2 &= -3\Sigma_3 \tag{3.22} \\
C_3 &= \frac{3}{2}C_{-1}\sum_{n=1}^{\infty}\frac{1}{n}\left(a_n\Sigma_2 a_n^\dagger + a_n^\dagger\Sigma_2 a_n - \Sigma_2\right) - \frac{15}{2}C_{-1}\left(\Sigma_2\right)^2 \tag{3.23} \\
C_4 &= \frac{-21}{2}\left(\Sigma_2\Sigma_3 + \Sigma_3\Sigma_2\right) + 3\sum_{n=1}^{\infty}\frac{1}{n}\left(a_n\Sigma_3 a_n^\dagger + a_n^\dagger\Sigma_3 a_n - \Sigma_3\right) \tag{3.24}
\end{align}
$$

The resulting Hamiltonian can be calculated perturbatively by inserting this expansion for a_0 in Eq. (2.9). We see that C_{-1}, C_1, and C_3 are even functions and therefore break the Z_2 symmetry. If we take the vacuum expectation value of a_0, we find

$$\langle 0|a_0|0\rangle = C_{-1}\left(\sqrt{-g} + \frac{3}{2}\zeta(2)\left(-g\right)^{-3/2} + \ldots\right) \, . \tag{3.25}$$

We see that for large negative g (small coupling) the VEV grows like $\sqrt{-g}$. The expansion breaks down at $g = 0$ and is not valid near the critical coupling. The second term in Eq. (3.25) is operator ordering sensitive and comes from the term $a_n\Sigma_2 a_n^\dagger$ in C_3. The Robertson calculation, for example, does not contain this second term. Finally we see that the expansion of a_0 contains only two and three point interactions as does the perturbative expansion in the symmetric phase. As we have argued for the symmetric phase, the perturbative expansion is tadpole free to all orders.

IV. Nonperturbative results

In I, we have shown analytically that if we truncate the problem to one mode, the zero mode constraint equation gives rise to critical behavior [1]. Subsequent numerical

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studies of the one-mode and the multi-mode problem in II have shown similar results. The term in the constraint equation that gives rise to spontaneous symmetry breaking is

$$\sum_{n>0} \frac{1}{n} \left(a_n a_0 a_n^\dagger + a_n^\dagger a_0 a_n \right) \ . \tag{4.1}$$

If we take matrix elements of this term we find that it couples the diagonal matrix elements of a_0 to all of the modes in the problem. This coupling of all of the length scales of the theory is the essential ingredient that generates spontaneous symmetry breaking.

This gives considerable insight into how spontaneous symmetry breaking will develop in QCD. For an operator to couple all length scales, it must be at least third order, and it must have the structure where there is a creation operator on one side of the zero-mode operator and a destruction operator on the other side. We conclude therefore that the fourth-order interaction in QCD is likely to be very important for spontaneous symmetry breaking.

In our previous work, I and II, we imposed a Tamm-Dancoff type truncation on the Fock space where we limited the number of modes M and the total number of allowed particles N. This allowed us to look at the various limits where the problem simplified. In this paper, we will introduce a cutoff in total momentum [6]. This cutoff is more consistent with the fact that the zero mode a_0 is block diagonal in states of equal P^+. This can also be considered a cutoff in the integer-valued "resolution" $K_{\max}$ with which we probe the theory [6]. We keep all states of the total longitudinal momentum P^+ such that

$$\frac{d}{2\pi} P^+ \leq K_{\max} \ . \tag{4.2}$$

As we go on to investigate this problem with finer and finer resolution we are able to analyze a number of issues that are unique to the broken phase of the problem near the critical coupling.

We have argued in the previous section that the standard mass subtraction, which has the form

$$- a_0 \sum_{n>0} \frac{1}{n} \ , \tag{4.3}$$

removes the divergence in Eq. (4.1) and gives a totally finite theory in perturbation theory, even for the broken phase. In the nonperturbative solution we do not calculate order by order as in perturbation theory, and such a cancellation is not assured. We

already saw indications in II that the critical coupling was growing with the number of modes in problem. In the numerical approach we use here this translates into a logarithmic divergence associated with the K_{max} cutoff.

For a finite resolution K_{max} the constraint equation (2.8) becomes a finite set of coupled cubic equations. These equations can then be solved numerically; a simple iterative method related to Newton's method was used to produce the results discussed here. To construct the iteration formulas, we write the matrix elements of the right-hand side of (2.8) as functions of g and the matrix elements of a_0

$$
\begin{aligned}
F_{\alpha,\beta}(g,a_0) &= \langle\alpha|\left\{ga_0 + a_0^3 + 2a_0\Sigma_2 + 2\Sigma_2 a_0\right. \\
&\left. + \sum_{n>0}\frac{1}{n}\left(a_n^\dagger a_0 a_n + a_n a_0 a_n^\dagger - a_0\right) + 6\Sigma_3\right\}|\beta\rangle ,
\end{aligned} \tag{4.4}
$$

where $|\alpha\rangle$ and $|\beta\rangle$ are generic Fock states allowed by the K_{max} cutoff. The constraint equation is then expressed as a set of algebraic equations

$$
F_{\alpha,\beta}(g,a_0) = 0 \tag{4.5}
$$

that determine all the matrix elements of a_0 as functions of g. However, in the following, we will treat $\langle 0|a_0|0\rangle$ as the independent variable and g as one of the dependent variables.

The nonlinear system (4.5) can be solved iteratively according to the following scheme:

$$
g^{(n+1)} = g^{(n)} - \omega\frac{F_{0,0}(g^{(n)}, a_0^{(n)})}{\langle 0|a_0|0\rangle} , \tag{4.6}
$$

$$
\langle\alpha|a_0^{(n+1)}|\beta\rangle = \langle\alpha|a_0^{(n)}|\beta\rangle - \omega\frac{F_{\alpha,\beta}(g^{(n)}, a_0^{(n)})}{\left.\frac{\partial F_{\alpha,\beta}(g^{(n)},a_0)}{\partial\langle\alpha|a_0|\beta\rangle}\right|_{a_0=a_0^{(n)}}} , \tag{4.7}
$$

where neither $|\alpha\rangle$ nor $|\beta\rangle$ is the vacuum state. The superscript (n) indicates the iteration number, and ω is an underrelaxation parameter that should usually be chosen to be between 0.5 and 0.75. This approach works well for values of $\langle 0|a_0|0\rangle$ much less than one and values of K_{max} less than 20, but otherwise converges slowly.

As part of the truncation process, assumptions must be made about the behavior of the zero-mode matrix elements for large momentum P^+ sectors. This is because each matrix element is coupled to an infinite sequence of other matrix elements through
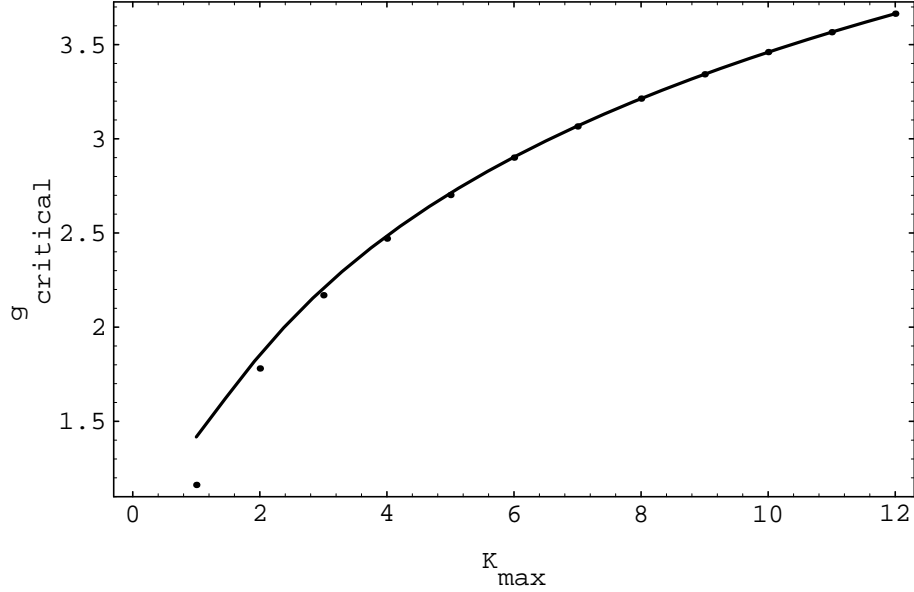
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Figure 5: Plot of g_{critical} vs K_{max} The solid line is the fit given in Eq. (4.8) of the text.

the sum that appears in the constraint equation (2.8). If one assumes that matrix elements decrease quickly to zero with increasing P^+ then the sum can be truncated at $n = K_{\text{max}}$; the remainder is guaranteed to be a small correction. This is what we have assumed in I and II.

Let us examine the asymptotic behavior of the zero-mode matrix elements more closely. We might expect that, in the limit of large n, the terms $\langle \alpha | (a_n^\dagger a_0 a_n + a_n a_0 a_n^\dagger) | \beta \rangle / n$ will cancel the term $-\langle \alpha | a_0 | \beta \rangle / n$ in the sum over n in the constraint equation (2.8). After all, this is why there are no tadpoles in perturbation theory. For the moment let us use this as our assumption for the Fock space truncation. For example, we will demand that the single-particle matrix element $\langle 0 | a_K a_0 a_K^\dagger | 0 \rangle$ be equal to $\langle 0 | a_0 | 0 \rangle$ for $K > K_{\text{max}}$. Unfortunately, if we examine the resulting numerical solutions, we find that the zero-mode matrix elements are inconsistent with this assumption and that there is a logarithmic divergence in g_{critical}, which is fit by the following form:

$$g_{\text{critical}} = 1.192 \ln K_{\text{max}} + 0.638 + \frac{0.779}{K_{\text{max}}} . \tag{4.8}$$

The values obtained are plotted in Fig. 5. This divergence appears to be due to the fact that the mass counterterm Eq. (4.3) does not cancel the divergence in Eq. (4.1). This behavior is illustrated in Fig. 6 for a calculation where $\langle 0 | a_0 | 0 \rangle = 0.05$; the
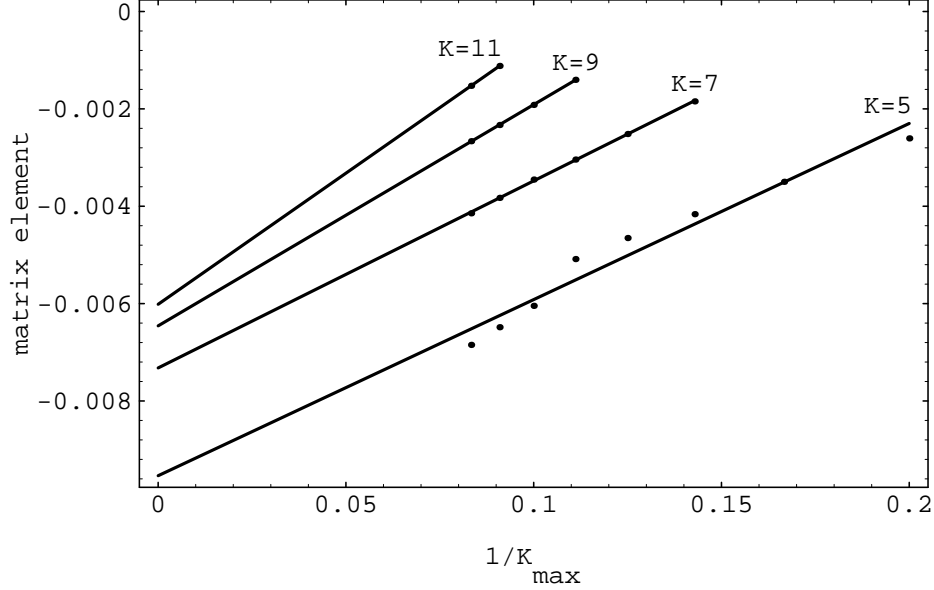
15

Figure 6: Single-particle matrix elements $\langle 0|a_K a_0 a_K^\dagger|0\rangle$ plotted for a series of values of the maximum resolution $K_{\max}$. The values of K selected are 5, 7, 9, and 11. The vacuum expectation value is set at 0.05.

asymptotic value of $\langle 0|a_K a_0 a_K^\dagger|0\rangle$ is between -0.003 to -0.006, which is certainly inconsistent with $\langle 0|a_0|0\rangle = 0.05$. The difference between the assumed and actual behaviors produces the logarithmic dependence on $K_{\max}$.

If the truncation of the system of equations is done with the assumption that the matrix elements tend to zero (as we did in I and II), then different results are obtained. The term $-a_0 \sum_n 1/n$ is immediately logarithmically divergent and guarantees a logarithmically divergent behavior for g_{critical}. However, the result for g_{critical}, shown in Fig. 7, has an additional logarithmic divergence. It is well fit by the form

$$g_{\text{critical}} = \sum_{n=1}^{K_{\max}} \frac{1}{n} + 0.214 \ln K_{\max} + 0.147 + \frac{0.0920}{K_{\max}} . \tag{4.9}$$

This is consistent with the results found in II and implies that the broken phase of the theory requires additional renormalization. Clearly, however, one cannot remove the logarithmic divergence in g by simply performing a different subtraction that would eliminate the $-a_0 \sum_n 1/n$ term in the constraint equation. The extra divergence comes from the behavior of matrix elements of a_0 in the limit of large momenta. Instead of going to zero, they tend toward a nonzero constant. As an illustration of
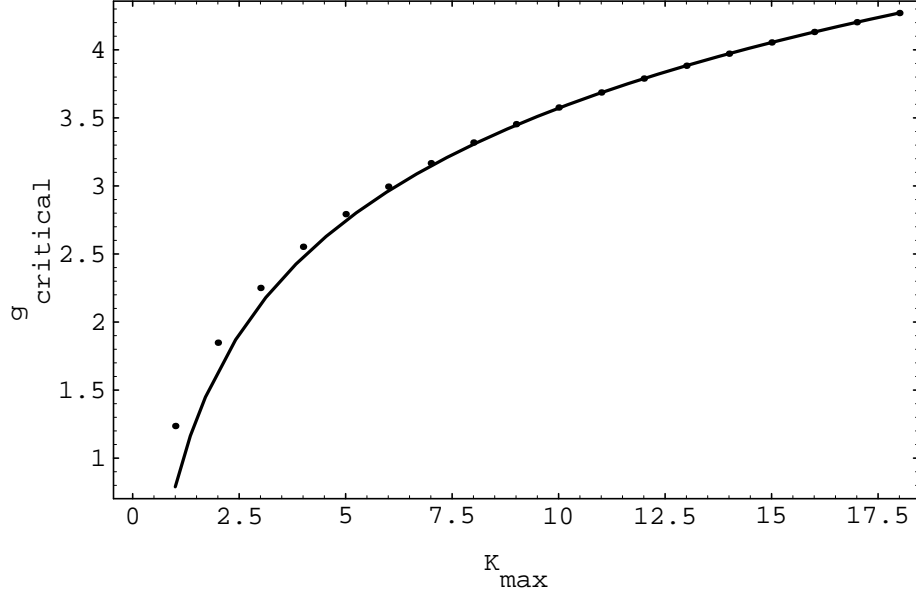
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Figure 7: Plot of g_{critical} vs K_{max} for the case where matrix elements are assumed to be asymptotically zero for large momentum. The solid line is the fit given in Eq. (4.9) of the text.
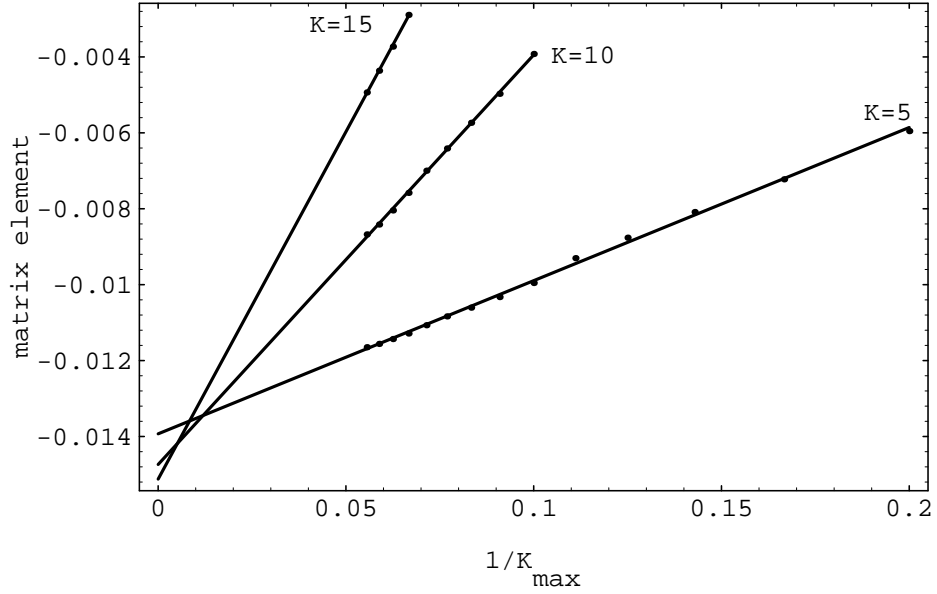

Figure 8: Single-particle matrix elements $\langle 0|a_K a_0 a_K^\dagger|0\rangle$ plotted for a series of values of the maximum resolution K_{max}. The values of K selected are 5, 10, and 15. The vacuum expectation value is set at 0.05. The matrix elements are calculated under the assumption that they are asymptotically zero.
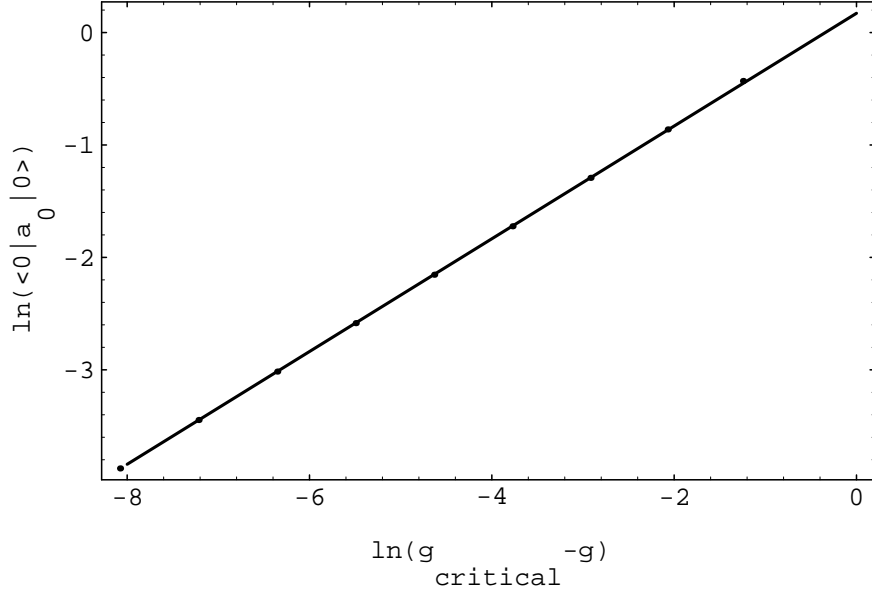
Figure 9: Plot of $\ln\left(\langle 0|a_0|0\rangle\right)$ vs $\ln\left(g_{\text{critical}} - g\right)$, with $K_{\text{max}} = 10$. The slope yields the critical exponent β, which appears in Eq. (4.10) of the text.

this, one-particle matrix elements are shown in Fig. 8. This behavior is inconsistent with the original assumption that the matrix elements go to zero in the limit of large momentum.

We can also investigate the shape of the critical curve near the critical coupling as we increase the resolution. The general theory of critical behavior indicates that the VEV should behave as a power law near the critical coupling [14]

$$\text{VEV} \propto \left(g_{\text{critical}} - g\right)^{\beta} . \tag{4.10}$$

We have calculated the exponent β as a function of the resolution, $K_{\text{max}} = 5$ to 12, and find it to be equal to 0.50, independent of K_{max}. A typical fit, where β is the slope, is shown in Fig. 9. However, $(\phi^4)_{1+1}$ is in the same universality class of theories as the Ising model, and for this model it is known that the exact value of β is $1/8$. Of course we would have been surprised if we obtained the exact answer near the critical point using an approximation method as primitive as DLCQ. Because we have studied the long range behavior of this theory with one mode of zero momentum, we should reasonably expect to obtain results that are about as good as those of a mean field calculation in the equal-time approach and, indeed, the value of the critical exponent that one obtains for the Ising model in the mean field approximation is $1/2$.

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V.   Conclusion

Our analysis of this rather simple two-dimensional scalar theory has produced a number of very interesting and important results for the program of solving QCD as a light-cone quantized field theory. The most important is that we can understand a variety of long-range phenomena starting from a simple Fock-space vacuum. We have found that the long-range physics of this theory is uniquely contained in a set of additional operators. In the DLCQ formulation these operators are determined, up to renormalizations, by an operator constraint equation (2.8). Furthermore, the theory that contains the new operators agrees with equal-time perturbation theory. Thus one is confident in this formulation that we are solving the theory that we set out to solve.

There are a number of ways in which this analysis is incomplete, and the reasons for this are important for our QCD program [15]. One is that we have not succeeded in renormalizing the new operators in the broken phase of the theory. The reason is that in two dimensions there are an infinite number of allowed operators. Thus the new operators introduced through the zero modes can be very complicated, and in fact *are* very complicated, making renormalization difficult. Unfortunately it has been shown that the number of allowed operators in QCD can be very large as well, and therefore we might find similar difficulties there.

Another way in which the analysis is incomplete is that the solutions to the constraint equation are not completely consistent with our Fock space truncation. This behavior is related to the logarithmic divergence of g_{critical} and is not well understood. It is unclear if similar problems will arise in a study of QCD.

A third deficiency is evident from the calculation of the critical exponent β. A complete analysis should yield $\beta = 1/8$ while we have obtained the mean-field value of $1/2$. Typically a mean field result indicates that one has not included enough length scales in the calculation. We used DLCQ as a regulator and also as a calculational technique: we chose our momenta on a linear scale in the numerical calculations. Thus, we were not able to include many length scales in the numerical calculations. We need to find a way of improving the numerical approach so that many length scales are included.

Acknowledgments

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